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Projectability of Left Invariant Nambu-Poisson Tensors on a Lie Group

by

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Dedicated to the memory of Professor Genji JIMBO

ABSTRACT. We study projectability of left invariant Nambu-Poisson tensors (LINPT) on a Lie group, and investigate the conditions for given Nambu-Poisson tensors to be projectable. Moreover we show that if η is an LINPT on G, which is projectable on an irreducible Riemannian symmetric space G/K, then η has only two possibilities.

1. Introduction

A Nambu-Poisson manifold is defined to be a pair of a C^∞-manifold and a *Nambu-Poisson tensor* defined on it [3], [5]. A Nambu-Poisson tensor is, by definition, a skew-symmetric contravariant tensor field on a manifold such that the induced bracket operation satisfies the *fundamental identity*, which is a generalization of the usual Jacobi identity.

Let G be a connected Lie group with left invariant volume form Ω, and K a connected closed subgroup of G. For a left invariant Nambu-Poisson tensor η on G, $\omega = i(\eta)\Omega$ is called a *Nambu-Poisson form*. This form is clearly left invariant. A left invariant Nambu-Poisson tensor η on G is said to be *projectable* if its corresponding Nambu-Poisson form ω is, in a usual sense, projectable on a G-invariant form $\bar{\omega}$ on G/K. In this article, we study projectable left invariant

Nambu-Poisson tensors η on a Lie group G. For example we will show that if G/K is an irreducible Riemannian symmetric space, then η has only two possibilities.

2. Reviews of Nambu-Poisson Manifolds

In this section, we will review some useful results of geometry of Nambu-Poisson manifolds. Details are referred to [3], [4]. Let M be an m-dimensional C^∞-manifold, and $\mathcal{F}$ its algebra of real valued C^∞ functions on M. We denote by $\Gamma(\Lambda^n TM)$ the space of global cross-sections $\eta : M \longrightarrow \Lambda^n TM$. Then for each $\eta \in \Gamma(\Lambda^n TM)$, there corresponds the bracket defined by

$$\{f_1, \dots, f_n\} = \eta\,(df_1, \dots, df_n), \quad f_1, \dots, f_n \in \mathcal{F}.$$

This bracket operation is an n-linear skew-symmetric map from $\mathcal{F}^n$ to $\mathcal{F}$ which satisfies the Leibniz rule:

$$\{f_1, \dots, f_{n-1}, g_1 \cdot g_2\} = \{f_1, \dots, f_{n-1}, g_1\} \cdot g_2 + g_1 \cdot \{f_1, \dots, f_{n-1}, g_2\},$$

for all $f_1, \dots, f_{n-1}, g_1, g_2 \in \mathcal{F}$.

Let $A = \sum f_{i_1} \wedge \cdots \wedge f_{i_{n-1}}, f_{i_j} \in \mathcal{F}$. Since the bracket operation clearly satisfies the Leibniz rule, we can define a vector field X_A corresponding to A by the following equation:

$$X_A(g) = \sum \{f_{i_1}, \dots, f_{i_{n-1}}, g\}, \quad g \in \mathcal{F}.$$

Such a vector field is called a *Hamiltonian vector field.* The space of Hamiltonian vector fields is denoted by $\mathcal{H}$.

Definition 2.1. *$\eta \in \Gamma(\Lambda^n TM)$ is called a Nambu-Poisson tensor of order n if it satisfies $\mathcal{L}(X_A)\eta = 0$ for all $X_A \in \mathcal{H}$, where $\mathcal{L}$ is the Lie derivative. Then a Nambu-Poisson manifold is a pair (M, η).*

The above definition is clearly equivalent to the following *fundamental identity:*

$$\begin{aligned}\{f_1, \dots, f_{n-1}, \{g_1, \dots, g_n\}\} = &\{\{f_1, \dots, f_{n-1}, g_1\}, g_2, \dots, g_n\} \\ &+ \{g_1, \{f_1, \dots, f_{n-1}, g_2\}, g_3, \dots, g_n\} \\ &+ \cdots + \{g_1, \dots, g_{n-1}, \{f_1, \dots, f_{n-1}, g_n\}\}\end{aligned}$$

for all $f_1, \ldots, f_{n-1}, g_1, \ldots, g_n \in \mathcal{F}$.

Let $\eta(p) \neq 0,\ p \in M$. Then we say that η is *regular* at p. Now we can state the following local structure theorem for Nambu-Poisson tensors [2], [3].

Theorem 2.1. *Let $\eta \in \Gamma(\Lambda^n TM),\ n \geq 3$. If η is a Nambu-Poisson tensor of order n, then for any regular point p, there exists a coordinate neighborhood U with local coordinates $(x_1, \ldots, x_n, x_{n+1}, \ldots, x_m)$ around p such that*

$$\eta = \frac{\partial}{\partial x_1} \wedge \cdots \wedge \frac{\partial}{\partial x_n}$$

on U, and vice versa.

To prove the above theorem, the condition $n \geq 3$ is essential. So whenever we mention a Nambu-Poisson manifold, we always assume that the order of the Nambu-Poisson tensor is greater than or equal to 3.

Let (M, η) be a Nambu-Poisson manifold with volume form Ω, and $m \geq n \geq 3$. Put $\omega = i(\eta)\Omega$, where the right hand side is the interior product of η and Ω. Hence ω is $(m-n)$-form. The following theorem gives a necessary and sufficient condition for η to be a Nambu-Poisson tensor. For the proof, see [4].

Theorem 2.2. *Let $\eta \in \Gamma(\Lambda^n TM)$, Then η is a Nambu-Poisson tensor if and only if η satisfies the following two conditions around each regular point:*

(a) *ω is (locally) decomposable, and*

(b) *there exists a locally defined 1-form θ such that $d\omega = \theta \wedge \omega$.*

Remark 2.1. It is clear that the above criterion for Nambu-Poisson tensors does not depend on the choice of volume form.

3. Left invariant Nambu-Poisson tensors on Lie groups

In this section, we consider left invariant Nambu-Poisson tensors (LINPT) on Lie groups. Let G be an m-dimensional connected Lie group, $m \geq 3$. Denote by $\mathfrak{g}$ the Lie algebra of left invariant vector fields on G. Using Theorem 2.1, we can easily obtain the following lemma [4].

Lemma 3.1. *Let η be a non-zero LINPT of order $n \geq 3$ on a Lie group G. Then η is globally decomposable. Namely there exist n elements $X_1, \ldots, X_n$ of $\mathfrak{g}$ such that η is written as $\eta = X_1 \wedge \cdots \wedge X_n$.*

By the above lemma, any LINPT η of order n can be written as a decomposable element of $\Lambda^n \mathfrak{g}$.

If a Lie subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ has a basis $\{X_1, \ldots, X_n\}$, $\mathfrak{h}$ is denoted by $\mathfrak{h} = <X_1, \ldots, X_n>$. The following theorem states that for every Lie subalgebra $\mathfrak{h}$, $n \geq 3$, there corresponds an LINPT of order dim $\mathfrak{h}$. (For the proof, see [4].)

Theorem 3.2. *Let G be an m-dimensional Lie group.*

(a) *Let $\mathfrak{h} = <X_1, \ldots, X_n>$ be an n-dimensional Lie subalgebra of $\mathfrak{g}$, $n \geq 3$. For basis $\{X_1, \ldots, X_n\}$ of $\mathfrak{h}$, put $\eta = X_1 \wedge \cdots \wedge X_n$. Then η is an LINPT of order n on G.*

(b) *Conversely given an LINPT $\eta = X_1 \wedge \cdots \wedge X_n \in \Lambda^n \mathfrak{g}$ on G, then $\mathfrak{h} = <X_1, \ldots, X_n>$ is an n-dimensional Lie subalgebra of $\mathfrak{g}$.*

If an LINPT η has two expressions: $\eta = X_1 \wedge \cdots \wedge X_n = Y_1 \wedge \cdots \wedge Y_n$, then we know that $<X_1, \ldots, X_n> = <Y_1, \ldots, Y_n>$. Thus we have

Corollary 3.3. *There is a one to one correspondence up to constant multiple between the set of LINPTs of order n on G and the set of n-dimensional Lie subalgebras of $\mathfrak{g}$.*

By Corollary 3.3, we know that there are many LINPTs on a Lie group. Hence, from now on, we shall consider LINPTs which can be projected down to some homogeneous space. Let η be an LINPT of order n and Ω a left invariant volume form on G. As in the previous section, put $\omega = i(\eta)\Omega$. Then ω is a left invariant $(m-n)$-form, which is called a left invariant Nambu-Poisson form (LINPF).

Let us fix our notations. Let G be an m-dimensional connected Lie group and K a k-dimensional connected closed subgroup of G. Denote by $\mathfrak{g}$ and $\mathfrak{k}$ the Lie algebras of G and K respectively. Let $\pi : G \longrightarrow G/K$ be the natural projection. The mapping $\bar{\gamma} \longrightarrow \pi^* \bar{\gamma}$ establishes a one to one correspondence between G-invariant p-forms $\bar{\gamma}$ on G/K and left invariant p-forms γ on G which satisfy

(1) $$i(X)\gamma = 0 \text{ for all } X \in \mathfrak{k},$$
(2) $$R_a^*\gamma = \gamma \text{ for all } a \in K\,[1].$$

Such a p-form γ is said to be *projectable.* If a p-form γ satisfies only the condition (1), γ is said to be *semi-projectable.* If K is connected, the condition (2) is replaced by the following equivalent condition.

(2)′ $$\mathcal{L}(X)\gamma = 0 \text{ for all } X \in \mathfrak{k}.$$

Definition 3.1. *An LINPT η of order n on G is said to be semi-projectable (resp. projectable) if the corresponding LINPF $\omega = i(\eta)\Omega$ is semi-projectable (resp. projectable) in the above sense.*

Let Ω and Ω' be any left invariant volume forms. Then $\Omega' = c\Omega$ for some non-zero constant c. Hence the above definition does not depend on the choice of left invariant volume forms.

In the rest of this section, we mainly consider a (semi-)projectable LINPT on G. Let $\mathfrak{g} = \langle X_1, \ldots, X_k, X_{k+1}, \ldots, X_m \rangle$ and $\mathfrak{k} = \langle X_1, \ldots, X_k \rangle$. Recall that each X_i is a left invariant vector field on G.

Lemma 3.4. *Let $\mathfrak{g} = \mathfrak{k} + \mathfrak{m}$, where $\mathfrak{m}$ is a complementary subspace of $\mathfrak{k}$ in $\mathfrak{g}$. Let η be a semi-projectable LINPT of order $l \geq 3$ on G. Then η has the following form:*

$$\eta = X_1 \wedge \cdots \wedge X_k \wedge F_1 \wedge \cdots \wedge F_{l-k},$$

where $X_i \in \mathfrak{k}$ and $F_j \in \mathfrak{m}$.

Proof. Since η is semi-projectable, we have

$$0 = i(X)\omega = i(X)i(\eta)\Omega = i(\eta \wedge X)\Omega,$$

for $X \in \mathfrak{k}$. Hence $\eta \wedge X = 0$ for any $X \in \mathfrak{k}$, and η is written as $\eta = X_1 \wedge \cdots \wedge X_k \wedge A$, where $A \in \Lambda^{l-k}\mathfrak{m}$. Due to Lemma 3.1, η is globally decomposable. Thus by an easy consideration, A is also decomposable and we obtain that η is written as

$$\eta = X_1 \wedge \cdots \wedge X_k \wedge F_1 \wedge \cdots \wedge F_{l-k},$$

where $X_i \in \mathfrak{k}$ and $F_j \in \mathfrak{m}$. □

Definition 3.2. *An LINPT η on G is said to be trivial with respect to the natural projection $G \longrightarrow G/K$ if η is equal to one of the following tensors up to constant multiple: $\eta = X_1 \wedge \cdots \wedge X_k$, or $\eta = X_1 \wedge \cdots \wedge X_k \wedge X_{k+1} \wedge \cdots \wedge X_m$.*

Let $\Omega = \omega_1 \wedge \cdots \wedge \omega_m$ be a left invariant volume form on G, where $\{\omega_i\}$ is the dual basis of $\{X_i\}$. If $\eta = X_1 \wedge \cdots \wedge X_k \wedge X_{k+1} \wedge \cdots \wedge X_m$, then $\omega = i(\eta)\Omega = 1$. Hence $d\omega = 0$. On the other hand, if $\eta = X_1 \wedge \cdots \wedge X_k$, then $\omega = i(\eta)\Omega = \omega_{k+1} \wedge \cdots \wedge \omega_m$. In general, this $(m-k)$-form ω is not always closed. For example, let $\mathfrak{g} = \mathfrak{sl}(3, R) = \mathfrak{a} + \mathfrak{n} + \mathfrak{k}$ be the usual Iwasawa decomposition. Let A and N be the connected Lie groups corresponding to $\mathfrak{a}$ and $\mathfrak{n}$ respectively. Then A and N are closed Lie subgroups of $SL(3, R)$. Put $\mathfrak{h} = \mathfrak{a} + \mathfrak{n}$. Denote by H the connected Lie group corresponding to $\mathfrak{h}$. H is diffeomorphic to $A \times N$ and hence H is a closed subgroup of $SL(3, R)$. Let us consider the natural projection $SL(3, R) \longrightarrow SL(3, R)/H$. We can find a basis $<X_1, \ldots X_8> = \mathfrak{g}$ such that $\mathfrak{a} = <X_1, X_2>$ and $\mathfrak{n} = <X_3, X_4, X_5>$. Put $\eta = X_1 \wedge \cdots \wedge X_5$. Then $\omega = i(\eta)\Omega = \omega_6 \wedge \omega_7 \wedge \omega_8$ with respect to the dual basis $\{\omega_1, \ldots, \omega_8\}$ of $\{X_1, \ldots, X_8\}$. We know that $i(\mathfrak{h})d\omega \neq 0$. Thus this LINPT η is trivial but is not projectable.

Next let us study the case that G/K are irreducible Riemannian symmetric spaces. Let $\mathfrak{g}$ be a semisimple Lie algebra with Cartan decomposition $\mathfrak{g} = \mathfrak{k} + \mathfrak{m}$. Put $\mathfrak{k} = <X_1, \ldots, X_k>$, $\mathfrak{m} = \{Y_1, \ldots, Y_q\}$, where $k + q = m = \dim G$. Let $\alpha_1, \ldots, \alpha_k$ (resp. $\beta_1, \ldots, \beta_q$) be the dual basis of $X_1, \ldots, X_k$ (resp. $Y_1, \ldots, Y_q$). Then $\Omega = \alpha_1 \wedge \cdots \wedge \alpha_k \wedge \beta_1 \wedge \cdots \wedge \beta_q$ is a left invariant volume form of G.

Theorem 3.5. *Let η be a semi-projectable LINPT of order $n \geq 3$ on G. If G/K is an irreducible Riemannian symmetric space, then η is trivial.*

Proof. By Lemma 3.4, we know that η can be written as $\eta = X_1 \wedge \cdots \wedge X_k \wedge F_1 \wedge \cdots \wedge F_{n-k}$, where $F_i \in \mathfrak{m}$. If $n - k = 0$, we have done. Suppose that $n - k \geq 1$. Put $\mathfrak{m}' = \{F_1, \ldots, F_{n-k}\}$, which is a subspace of $\mathfrak{m}$. Recall that $[\mathfrak{m}, \mathfrak{m}] \subset \mathfrak{k}$ and $[\mathfrak{k}, \mathfrak{m}] \subset \mathfrak{m}$. By Theorem 3.2, $\mathfrak{k} + \mathfrak{m}'$ is a Lie subalgebra of $\mathfrak{g}$. Hence $[\mathfrak{k}, \mathfrak{m}'] \subset \mathfrak{m}'$. Thus we obtain

$$[[\mathfrak{m}, \mathfrak{m}], \mathfrak{m}'] \subset [\mathfrak{k}, \mathfrak{m}'] \subset \mathfrak{m}'.$$

Since $[\mathfrak{m}, \mathfrak{m}]$ acts irreducibly on $\mathfrak{m}$, we have $\mathfrak{m}' = \mathfrak{m}$. This implies that η is trivial. □

Remark 3.1. As is well-known, a G-invariant form $\bar{\omega}$ on a symmetric space G/K is always closed, and hence $\omega = \pi^*\bar{\omega}$ is also closed. Thus in this case, for a semi-projectable LINPT η, η is projectable if and only if $d\omega = 0$.

We give an example of non trivial LINPTs which are projectable on a *reducible* symmetric space. Put $G = SO(4)$ and $K = SO(2) \times SO(2)$. Then G/K is a reducible symmetric space which is locally diffeomorphic to $S^2 \times S^2$. One can find a basis $\mathfrak{o}(4) = \langle X_1, \ldots, X_6 \rangle$ which satisfies:

$$
\begin{aligned}
&[X_1, X_2] = -X_4, \quad [X_1, X_3] = -X_5, \quad [X_1, X_4] = X_2, \\
&[X_1, X_5] = X_3, \quad [X_2, X_3] = -X_6, \quad [X_2, X_4] = -X_1, \\
&[X_2, X_6] = X_3, \quad [X_3, X_5] = -X_1, \quad [X_3, X_6] = -X_2, \\
&[X_4, X_5] = -X_6, \quad [X_4, X_6] = X_5, \quad [X_5, X_6] = -X_4, \\
&[X_1, X_6] = [X_2, X_5] = [X_3, X_4] = 0.
\end{aligned}
$$

With respect to this basis, $\mathfrak{o}(2) \times \mathfrak{o}(2) = \langle X_1, X_6 \rangle$. Let $\{\omega_1, \ldots, \omega_6\}$ be the dual basis. Then we can find three projectable LINPTs:

$$
\begin{aligned}
\eta_1 &= X_1 \wedge X_2 \wedge X_3 \wedge X_4 \wedge X_5 \wedge X_6, \\
\eta_2 &= X_1 \wedge (X_2 + X_5) \wedge (X_3 - X_4) \wedge X_6, \\
\eta_3 &= X_1 \wedge (X_3 + X_4) \wedge (X_2 - X_5) \wedge X_6.
\end{aligned}
$$

In our definitions, the order of Nambu-Poisson tensors is greater than 2. So $\eta_4 = X_1 \wedge X_6$ is not an LINPT but a regular Poisson tensor on $SO(4)$. It is easy to see that η_2 and η_3 are non trivial projectable LINPTs. Put $\theta_i = i(\eta_i)\Omega$, where $\Omega = \omega_1 \wedge \cdots \wedge \omega_6$. Then we have $\theta_2 = (\omega_2 - \omega_5) \wedge (\omega_3 + \omega_4)$, and $\theta_3 = (\omega_3 - \omega_4) \wedge (\omega_2 + \omega_5)$. An easy computation shows that $d\theta_2 = d\theta_3 = 0$. Moreover θ_2 and θ_3 are cohomologous on $SO(4)$. Since θ_2 and θ_3 are projectable 2-forms, they are considered to be the forms on $SO(4)/SO(2) \times SO(2)$. Since $\theta_2 - \theta_3 = d(2\omega_6)$ and ω_6 can not be considered as a form on G/K, they are *not* cohomologous as 2-forms on $SO(4)/SO(2) \times SO(2)$. And hence they become generators of $H^2(SO(4)/SO(2) \times SO(2))$.

Let us consider the case that G/K is not a symmetric space. For example, let $G = U(n+1)$ and $K = U(n)$. Then G/K is not a symmetric space but a $(2n+1)$-dimensional homogeneous space which is diffeomorphic to S^{2n+1}. The Lie algebra $\mathfrak{u}(n)$ of $U(n)$ is contained in the Lie algebra $\mathfrak{u}(n+1)$ of $U(n+1)$ in the natural manner. Let $X_1, \dots, X_{n^2}$ be a basis of $\mathfrak{u}(n)$. Define matrices $Y_{2i-1} = (a_{pq})$ and $Y_{2i} = (b_{pq})$ for $1 \leq i \leq n$ and $Y_{2n+1} = (c_{pq})$ of $\mathfrak{u}(n+1)$ by

$$a_{i,n+1} = 1, \quad a_{n+1,i} = -1, \quad \text{otherwise } 0,$$
$$b_{i,n+1} = \sqrt{-1}, \quad b_{n+1,i} = \sqrt{-1}, \quad \text{otherwise } 0,$$
$$c_{n+1,n+1} = \sqrt{-1}, \quad \text{otherwise } 0.$$

Then $\mathfrak{u}(n+1) = <X_1, \dots, X_{n^2}, Y_1, \dots, Y_{2n+1}>$. Let $\{\alpha_1, \dots, \alpha_{n^2}, \beta_1, \dots, \beta_{2n+1}\}$ be its dual basis. Under these notations, define η by

$$\eta = X_1 \wedge \cdots \wedge X_{n^2} \wedge Y_{2n+1}.$$

Then $<X_1, \dots, X_{n^2}, Y_{2n+1}>$ is a Lie subalgebra of $\mathfrak{u}(n+1)$. With respect to the volume form $\Omega = \alpha_1 \wedge \cdots \wedge \alpha_{n^2} \wedge \beta_1 \wedge \cdots \wedge \beta_{2n+1}$, $\omega = i(\eta)\Omega = \beta_1 \wedge \cdots \wedge \beta_{2n}$ is a $2n$-form which is projected down to G/K. Since $d\omega = 0$, we find that η is a nontrivial projectable LINPT.

Next let us consider the conditions for a left invariant Nambu-Poisson form (LINPF) ω to be a closed form. Let G be an m-dimensional connected Lie group with Lie algebra $\mathfrak{g}$, and K be a k-dimensional connected closed Lie subgroup of G with Lie subalgebra $\mathfrak{k} = <X_1, \dots, X_k>$. With respect to the canonical projection $\pi : G \longrightarrow G/K$, we consider a projectable LINPT η of order $l (\geq k)$ on G. Then by Lemma 3.1, we know that η has the expression: $\eta = X_1 \wedge \cdots \wedge X_k \wedge X_{k+1} \wedge \cdots \wedge X_l$. (See also the proof of Lemma 3.4.) Adding $(m-l)$ vector fields $X_{l+1}, \dots, X_m$ to $X_1, \dots, X_l$, we make a basis of $\mathfrak{g}$. Recall that each vector field X_i, $(1 \leq i \leq m)$ is left invariant. We denote the dual basis of $<X_1, \dots, X_m>$ by $\omega_1, \dots, \omega_m$, and put $\Omega = \omega_1 \wedge \cdots \wedge \omega_m$. Then Ω is a left invariant volume form on G. An LINPF ω corresponding to η is given by $\omega = i(\eta)\Omega = \omega_{l+1} \wedge \cdots \wedge \omega_m$.

Proposition 3.6. *Let η be a semi-projectable LINPT of order l on G. Let $<X_1, \dots, X_m>$ be the basis of $\mathfrak{g}$ constructed from η as above, and let $\{C^k_{ij}\}$ be structure constants of $\mathfrak{g}$ corresponding to the basis $<X_1, \dots, X_m>$. If $\sum_{p=l+1}^{m} C^p_{rp} = 0$ for each r, $1 \leq r \leq l$, then η is a projectable LINPT.*

Proof. Since η is semi-projectable, η can be written as $\eta = X_1 \wedge \cdots \wedge X_k \wedge X_{k+1} \wedge \cdots \wedge X_l$. Then $<X_1, \dots, X_k, X_{k+1}, \dots, X_l>$ is a Lie subalgebra of $\mathfrak{g}$, and we easily obtain

$$d\omega = -\sum_{r=1}^{l}\sum_{p=l+1}^{m} C_{rp}^{p}\omega_r \wedge \omega .$$

Thus the condition $\sum_{p=l+1}^{m} C_{rp}^{p} = 0,\ 1 \leq r \leq l$ implies $d\omega = 0$. Then for any $X \in \mathfrak{k}$,

$$\mathcal{L}(X)\omega = i(X)d\omega + di(X)\omega = 0,$$

and hence η is projectable. □

Let us apply Proposition 3.6 to the case that G is a connected unimodular Lie group. Under this condition, we have

Corollary 3.7. *Let $\eta = X_1 \wedge \cdots \wedge X_l$ be a semi-projectable LINPT on a connected unimodular Lie group G. If the connected Lie subgroup L corresponding to the Lie algebra $\mathfrak{l} = \langle X_1, \ldots, X_l \rangle$ is also unimodular, then the LINPF $\omega = i(\eta)\Omega$ is closed, and hence η is projectable.*

Proof. Since G and L are unimodular, their structure constants satisfy $\sum_{\alpha=1}^{m} C_{i\alpha}^{\alpha} = 0$ for each i, $1 \leq i \leq m$, and $\sum_{\beta=1}^{l} C_{j\beta}^{\beta} = 0$ for each j, $1 \leq j \leq l$. Hence $\sum_{p=l+1}^{m} C_{rp}^{p} = 0$ for each r, $1 \leq r \leq l$. Due to Proposition 3.6, this implies $d\omega = 0$, and η becomes projectable. □

A typical example of Corollary 3.7 is the case that $G = SO(n)$ (resp. $U(n)$) and $K = SO(q)$ (resp. $U(q)$). Then G and K are unimodular Lie groups, and G/K is a Stiefel manifold. Let η be a semi-projectable LINPT on G whose corresponding Lie algebra induces a closed Lie subgroup L of G. Then $SO(q) \subset L \subset SO(n)$ (or $U(q) \subset L \subset U(n)$). Since L is a closed Lie subgroup of G, L is unimodular. Thus by Corollary 3.7, the LINPF $\omega = i(\eta)\Omega$ is closed and η is projectable.

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